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On the Conditional Variance for Scale Mixtures of Normal Distributions

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For a scale mixture of normal vector, $\mathbf{X}=A^{1/2}\mathbf{G}$, where $\mathbf{X}$, $\mathbf{G}\in\mathbb{R}^n$ and A is a positive variable, independent of the normal vector $\mathbf{G}$, we obtain that the conditional variance covariance, $\mathrm{Cov}(\mathbf{X}_2 \mid \mathbf{X}_1)$, is always finite a.s. for $m\geqslant 2$, where $\mathbf{X}_1\in\mathbb{R}^n$ and $m<n$, and remains a.s. finite even for $m=1$, if and only if the square root moment of the scale factor is finite. It is shown that the variance is not degenerate as in the Gaussian case, but depends upon a function $S_{A,m}(\cdot)$ for which various properties are derived. Application to a uniform and stable scale of normal distributions are also given. © 2000 Academic Press

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1. INTRODUCTION

The distribution of an n-dimensional random vector (column) $\mathbf{X}$ is a scale mixture of a normal distribution, if $\mathbf{X}:\stackrel{d}{=}A^{1/2}\mathbf{G}$, where A is a positive random variable independent of the n-dimensional Gaussian random (column) vector $\mathbf{G}$ with mean 0 and positive definite covariance matrix Σ, and the equality is in distribution.

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Gupta and Huang (1981) characterized scale mixtures (variance mixtures) of normal distributions by showing an equivalence of this class and complete monotonicity property on $(0, \infty)$. Bearing this property, it was found that this family includes the Cauchy, Laplace, student's t, symmetric stable (these were also found by Kelker, 1971), logistic, and double exponential distributions (Andrews and Mallows, 1974). Schoenberg (1938), Crawford (1977), and Miciewicz and Scheffer (1990) characterized this family by showing that if $\mathbf{X}(\mathbf{X} \in \mathbb{R}^n, n \geqslant 2)$ is scale mixture of multivariate normal distribution, then its characteristic function, $\varphi_{\mathbf{X}}(\mathbf{t}), \mathbf{t} \in \mathbb{R}^n$, has the representation $\varphi_{\mathbf{X}}(\mathbf{t}) = \psi(\|\mathbf{t}^2\|)$, where $\|\cdot\|$ denotes the Euclidean distance, and ψ denotes some function on $(0, \infty)$. It should be added here that the family discussed by Schoenberg (1938), Crawford (1977), and Miciewicz and Scheffer (1990) is much broader than the family of scale mixtures of normal distributions. Keilson and Steutel (1974) characterized this family in terms of moment existence. It can be shown that $E[A^p] < \infty$, if and only if $E[\|\mathbf{X}\|^p] < \infty$ for some $p > 0$; for example, if A is distributed as gamma, beta, or uniform then $E[\|\mathbf{X}\|^p] < \infty, \forall p > 0$. However, if A is totally right skewed $\alpha/2$-stable, $0 < \alpha < 2$, with Laplace transform $E[\exp(-uA)] = \exp(-u^{a/2})$, $u \geqslant 0$ then $E[A^p] < \infty$, if and only if $p < a/2$. In this case, $\mathbf{X}$ has a multivariate symmetric a-stable distribution, and $E[\prod_{i=1}^n |X_i|^{p_i}] < \infty$, for $p_i \geqslant 0$, $i = 1, ..., n$, and $\sum_{i=1}^n p_i = p < a$ (Samorodnitsky and Taqqu, 1990). Thus, their second moment is always infinite and so is their first absolute moment when $0 < \alpha \leqslant 1$.

Here, we are interested in conditional variances, and these may exist and be finite even when their unconditional counterparts are infinite. For $1 \leqslant m < n$, we will write $\mathbf{X} = (\mathbf{X}_1, \mathbf{X}_2)$, $\mathbf{G} = (\mathbf{G}_1, \mathbf{G}_2)$ and

$$\Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix},$$

where $\mathbf{X}_1$ and $\mathbf{G}_1$ are m-dimensional and Σ_{11} is $m \times m$-dimensional, i.e., Σ_{11} is the covariance matrix of $\mathbf{G}_1$, etc. The conditional distribution of $\mathbf{G}_2$ given $\mathbf{G}_1$ is normal with mean $\Sigma_{21} \Sigma_{11}^{-1} G_1$ and covariance matrix $\Sigma_{22} - \Sigma_{21} \Sigma_{11}^{-1} \Sigma_{12}$; i.e., the conditional mean of $\mathbf{G}_2$ given $\mathbf{G}_1$ depends linearly on $\mathbf{G}_1$ and the conditional variance–covariance of $\mathbf{G}_2$ given $\mathbf{G}_1$ is constant (degenerate, non random) and does not depend on the value of $\mathbf{G}_1$:

$$E[\mathbf{G}_2 \mid \mathbf{G}_1] = \Sigma_{21} \Sigma_{11}^{-1} \mathbf{G}_1, \qquad \mathrm{Cov}(\mathbf{G}_2 \mid \mathbf{G}_1) = \Sigma_{22} - \Sigma_{21} \Sigma_{11}^{-1} \Sigma_{12} := \Sigma_{2|1}. \tag{1.1}$$

This is the archetypical homoscedastic example, where regressions are linear and conditional variances constant. The regression theory has been extended beyond the normal theory. Hardin (1982) considers that **X** is a symmetric stable random vector, and he claims that **X** is spherically generated if and only if **X** is a scale mixture of multivariate normal, where the scale random variable is stable totally skewed to the right (sub-Gaussian). He continued in showing that a symmetric stable random vector **X**, with $\dim(sp(\mathbf{X})) \geqslant 3$, having the linear regression property must be a sub-Gaussian vector. This result, coupled with the fact that any stable random vector with $\dim(sp(\mathbf{X})) < 3$, has the linear regression property. This property agrees with (1.1), which is related to the normal theory. The disagreement, however, occurs when one looks at how the conditional variance–covariance behaves. It will be shown that scale mixtures of normal distributions do not have constant conditional variances, so they provide heteroscedastic examples, and we will examine these non-linear conditional functions.

This article is structured as follows. Section 2 presents the main results with their proofs. Section 3 demonstrates how to apply some of these results to uniform and stable cases. Section 4 gives the proofs of some of the secondary results. The auxiliary results are displayed in Section 5.

2. THE RESULTS

Our first result shows that the conditional second moment of each component of $\mathbf{X}_2$ given $\mathbf{X}_1$ is always finite when the dimensionality of $\mathbf{X}_1$ is two or more. Furthermore, we find a necessary and sufficient condition when $\mathbf{X}_1$ is univariate, and we express the conditional covariance matrix of $\mathbf{X}_2$ given $\mathbf{X}_1$ (under appropriate conditions) in terms of the distribution and the Laplace transform of A.

THEOREM 1. I. *The conditional second moment of the components of* $\mathbf{X}_2$ *given* $\mathbf{X}_1$ *is finite a.s. always when* $m \geqslant 2$ *and if and only if* $E[A^{1/2}] < \infty$ *when* $m = 1$.

II. *If* $m \geqslant 2$, *or if* $m = 1$ *and* $E[A^{1/2}] < \infty$, *then*

$$\operatorname{Cov}(\mathbf{X}_2 \mid \mathbf{X}_1) = \Sigma_{2\mid 1} S^2_{A,m}((\mathbf{X}_1' \Sigma_{11}^{-1} \mathbf{X}_1)^{1/2}) \qquad a.s. \tag{2.1}$$

where

$$S^2_{A,m}(x) = \frac{\int_{[0,\infty)} u^{-m/2+1} \exp\left(-\frac{x^2}{2u}\right) dF_A(u)}{\int_{[0,\infty)} u^{-m/2} \exp\left(-\frac{x^2}{2u}\right) dF_A(u)}, \qquad x \geqslant 0. \tag{2.2}$$

III. *If the Laplace transform L_A of A satisfies*

$$\int_{[0,\infty)} u^{m/2-1} L_A(u)\,du < \infty \qquad \text{and} \qquad \int_{[0,\infty)} u^{m/2-1} L'_A(u)\,du < \infty, \tag{2.3}$$

then (2.2) *holds and* $S^2_{A,m}(x)$, $x \geqslant 0$, *can be expressed as*

$$S^2_{A,1}(x) = \frac{-\int_0^\infty L'_A(r^2)\cos(\sqrt{2}\,xr)\,dr}{\int_0^\infty L_A(r^2)\cos(\sqrt{2}\,xr)\,dr} \tag{2.4}$$

and, for $m \geqslant 2$,

$$S^2_{A,m}(x) = \frac{\int_0^\infty r^{m/2} L'_A(r^2)\, J_{(m-2)/2}(\sqrt{2}\,xr)\,dr}{\int_0^\infty r^{m/2} L_A(r^2)\, J_{(m-2)/2}(\sqrt{2}\,xr)\,dr}, \tag{2.5}$$

where $J_v(\cdot)$ *is the Bessel function of the first kind with* $v > 0$.

Proof. I. To demonstrate the proof of this theorem, we reiterate some of the classical results of normal theory. For simplicity of notation, it suffices to consider the case where $n = m+1$, so X_2, Σ_{22} are scalar. Then

$$E[X_2^2 \mid \mathbf{X}_1] = E[E[AG_2^2 \mid A, \mathbf{G}_1] \mid \mathbf{X}_1] = E[AE[G_2^2 \mid \mathbf{G}_1] \mid \mathbf{X}_1],$$

and since

$$E[G_2^2 \mid \mathbf{G}_1] = \sigma_2^2 - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12} + E^2[G_2 \mid \mathbf{G}_1] = s_2^2 + (\Sigma_{21}\Sigma_{11}^{-1}\mathbf{G}_1)^2,$$

where $s_2^2 = \sigma_2^2 - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12}$, we have

$$E[X_2^2 \mid \mathbf{X}_1 = \mathbf{x}_1] = s_2^2 E[A \mid \mathbf{X}_1 = \mathbf{x}_1] + (\Sigma_{21}\Sigma_{11}^{-1}\mathbf{x}_1)^2. \tag{2.6}$$

It follows that $E[X_2^2 \mid \mathbf{X}_1] < \infty$ a.s. if and only if $E[A \mid \mathbf{X}_1] < \infty$ a.s. and by Proposition 1 in Section 4, if and only if

$$\int_{[0,\infty)} u^{-m/2+1} \exp\left(-\frac{1}{2u}\mathbf{X}'_1\Sigma_{11}\mathbf{X}_1\right) dF_A(u) < \infty \qquad \text{a.s.} \tag{2.7}$$

Note that for each fixed value of $\mathbf{X}_1$, the integrand is a continuous function of u over $(0,\infty)$, and tends to 0 as $u \downarrow 0$ and as $u \uparrow \infty$ if $m \geqslant 2$ and is bounded by $u^{1/2}$ if $m = 1$. Hence the conditional second moment is finite when $m \geqslant 2$ and when $m = 1$ is finite if and only if

$$\int_0^\infty u^{1/2}\,dF_A(u) < \infty \qquad \text{or} \quad E[A^{1/2}] < \infty.$$

II. We have

$$E[\mathbf{X}_2\mathbf{X}_2' \mid \mathbf{X}_1] = E[E[A\mathbf{G}_2\mathbf{G}_2' \mid A, \mathbf{G}_1] \mid \mathbf{X}_1] = E[AE[\mathbf{G}_2\mathbf{G}_2' \mid \mathbf{G}_1] \mid \mathbf{X}_1],$$

and since

$$\begin{aligned} E[\mathbf{G}_2\mathbf{G}_2' \mid \mathbf{G}_1] &= \Sigma_{2|1} + E[\mathbf{G}_2 \mid \mathbf{G}_1]\, E[\mathbf{G}_2' \mid \mathbf{G}_1] \\ &= \Sigma_{2\,|\,1} + \Sigma_{21}\Sigma_{11}^{-1}\mathbf{G}_1\mathbf{G}_1'\Sigma_{11}^{-1}\Sigma_{21}', \end{aligned} \tag{2.8}$$

and using the conditional expectation it follows that

$$\begin{aligned} E[\mathbf{X}_2\mathbf{X}_2' \mid \mathbf{X}_1] &= \Sigma_{2|1} E[A \mid \mathbf{X}_1] + \Sigma_{21}\Sigma_{11}^{-1}\mathbf{X}_1\mathbf{X}_1'\Sigma_{11}^{-1}\Sigma_{21}' \\ &= \Sigma_{2\,|\,1} E[A \mid \mathbf{X}_1] + E[\mathbf{X}_2 \mid \mathbf{X}_1]\, E[\mathbf{X}_2' \mid \mathbf{X}_1]. \end{aligned}$$

Thus the covariance is given by

$$\begin{aligned} \operatorname{Cov}(\mathbf{X}_2 \mid \mathbf{X}_1) &= E[\mathbf{X}_2\mathbf{X}_2' \mid \mathbf{X}_1] \\ &\quad - E[\mathbf{X}_2 \mid \mathbf{X}_1]\, E[\mathbf{X}_2' \mid \mathbf{X}_1] = \Sigma_{2\,|\,1} E[A \mid \mathbf{X}_1], \end{aligned}$$

and by Proposition 1, $E[A \mid \mathbf{X}_1] = S^2_{A,m}((\mathbf{X}_1'\Sigma_{11}^{-1}\mathbf{X}_1)^{1/2})$ with $S^2_{A,m}(x)$ as in Theorem 1.II.

III. For every $u \geqslant 0$ and (column) vector $\mathbf{t} \in \mathbb{R}^m$, we have

$$\begin{aligned} E[\exp(-uA + i\mathbf{t}'\mathbf{X}_1)] &= E[E[\exp(-uA + iA^{1/2}\mathbf{t}'\mathbf{G}_1) \mid A]] \\ &= E[\exp(-uA - \tfrac{1}{2}A\mathbf{t}'\Sigma_{11}\mathbf{t})] = L_A(u + \tfrac{1}{2}\mathbf{t}'\Sigma_{11}\mathbf{t}). \end{aligned} \tag{2.9}$$

Putting $u = 0$ we obtain

$$E[\exp(i\mathbf{t}'\mathbf{X}_1)] = L_A(\tfrac{1}{2}\mathbf{t}'\Sigma_{11}\mathbf{t}),$$

and since the right-hand side is an integrable function of t over $\mathbb{R}^m$, in view of (2.4) we obtain

$$\begin{aligned} \int_{\mathbb{R}^m} L_A(\tfrac{1}{2}\mathbf{t}'\Sigma_{11}\mathbf{t})\, d\mathbf{t} &= (\det \Sigma_{11})^{-1/2} \int_{\mathbb{R}^m} L_A(\tfrac{1}{2}\mathbf{s}'\mathbf{s})\, d\mathbf{s})\, d\mathbf{s}, \qquad (\mathbf{s} = \Sigma_{11}^{1/2}\mathbf{t}) \\ &= \text{const} \int_0^\infty L_A(\tfrac{1}{2}r^2)\, r^{m-1}\, dr \qquad \text{(in polar coordinates)} \\ &= \text{const} \int_0^\infty L_A(u)\, u^{m/2-1}\, du < \infty. \end{aligned} \tag{2.10}$$

By the inversion of the Fourier transform, we conclude that

$$f_{\mathbf{x}_1}(\mathbf{x}_1) = \frac{1}{(2\pi)^m} \int_{\mathbb{R}^m} e^{i\mathbf{t}'\mathbf{x}_1} L_A\left(\frac{1}{2}\,\mathbf{t}'\Sigma_{11}\mathbf{t}\right) d\mathbf{t}. \tag{2.11}$$

Now differentiating both sides of (2.9) with respect to $u > 0$, we obtain

$$-E[E[Ae^{-uA} \mid \mathbf{X}_1 = \mathbf{x}_1]\, e^{i\mathbf{t}'\mathbf{x}_1}] = L'_A(u + \tfrac{1}{2}\mathbf{t}'\Sigma_{11}\mathbf{t}).$$

Since $L_A(\cdot)$ is completely monotone on $(0, \infty)$, i.e., $(-1)^n\, L_A^{(n)}(u) \geqslant 0$, for $u > 0$, it follows that $-L'_A(u + \frac{1}{2}\mathbf{t}'\Sigma_{11}\mathbf{t}) \leqslant -L'_A(\frac{1}{2}\mathbf{t}'\Sigma_{11}\mathbf{t}) \in \mathbf{L}^1(\mathbb{R}^m)$. By (2.10) and (2.2), inversion of the Fourier transform yields

$$\begin{aligned} &-E[Ae^{-uA} \mid \mathbf{X}_1 = \mathbf{x}_1]\, f_{\mathbf{X}_1}(\mathbf{x}_1) \\ &= \frac{1}{(2\pi)^m} \int_{\mathbb{R}^m} e^{i\mathbf{t}'\mathbf{x}_1} L'_A\left(u + \frac{1}{2}\,\mathbf{t}'\Sigma_{11}\mathbf{t}\right) d\mathbf{t}, \qquad \text{a.e. in} \quad \mathbf{x}_1 \in \mathbb{R}^m, \end{aligned} \tag{2.12}$$

for each fixed $u > 0$. Since $f_{\mathbf{X}_1}(\mathbf{x}_1)$ and the right-hand side are continuous functions of $\mathbf{x}_1$ by III., and in (1.4) we consider the regular version of $E[Ae^{-uA} \mid \mathbf{X}_1 = \mathbf{x}_1]$, which is defined by (2.12) for all $u > 0$ and $\mathbf{x}_1 \in \mathbb{R}^m$. Now, letting $u \downarrow 0$ in (2.12) we obtain

$$E[A \mid \mathbf{X}_1 = \mathbf{x}_1]\, f_{\mathbf{x}_1}(\mathbf{x}_1) = \frac{1}{(2\pi)^m} \int_{\mathbb{R}^m} e^{i\mathbf{t}'\mathbf{x}_1} L'_A\left(\frac{1}{2}\,\mathbf{t}'\Sigma_{11}\mathbf{t}\right) d\mathbf{t}, \tag{2.13}$$

since the left-hand side of (2.12) converges pointwise to the left hand side of (2.13), and likewise for the right-hand side by dominated convergence theorem, since $L_A(u) = E[e^{-uA}]$ implies $L'_A(u) = -E[Ae^{-uA}]$ and for all $v > 0$, $-L'_A(u + v) = E[Ae^{-(u+v)A}] \to E[Ae^{-vA}] = -L'_A(v)$, as $u \downarrow 0$, and $L'_A(\frac{1}{2}\mathbf{t}'\Sigma_{11}\mathbf{t}) \in \mathbb{L}^1(\mathbb{R}^m)$ by (1.4) and (2.10). From (2.11) and (2.13) we obtain

$$S^2_{A,m}((\mathbf{x}'_1\Sigma_{11}^{-1}\mathbf{x}_1)^{1/2}) = E[A \mid \mathbf{X}_1 = \mathbf{x}_1] = \frac{-\int_{\mathbb{R}^m} e^{-i\mathbf{t}'\mathbf{x}_1} L'_A(\frac{1}{2}\mathbf{t}'\Sigma_{11}\mathbf{t})\, d\mathbf{t}}{\int_{\mathbb{R}^m} e^{i\mathbf{t}'\mathbf{x}_1} L_A(\frac{1}{2}\mathbf{t}'\Sigma_{11}\mathbf{t})\, d\mathbf{t}}. \tag{2.14}$$

We will now evaluate more explicitly the integrals appearing in the numerator and denominator. Putting $B = 2^{-1/2}\Sigma_{11}^{1/2}$ and $\mathbf{y} = B\mathbf{t}$, We have $\frac{1}{2}\mathbf{t}'\Sigma_{11}\mathbf{t} = \mathbf{t}'B' = \mathbf{y}'\mathbf{y} = \|\mathbf{y}\|^2$ and

$$\begin{aligned} F_m((\mathbf{x}'_1\Sigma_{11}\mathbf{x}_1)^{1/2}) &:= \int_{\mathbb{R}^m} e^{-i\mathbf{t}'\mathbf{x}_1} f(\tfrac{1}{2}\mathbf{t}'\Sigma_{11}\mathbf{t})\, d\mathbf{t} \\ &= (\det B)^{-1} \int_{\mathbb{R}^m} e^{i\mathbf{x}'_1 B^{-1}\mathbf{y}} f(\|\mathbf{y}\|^2)\, d\mathbf{y}. \end{aligned} \tag{2.15}$$

Going to polar coordinates $\mathbf{y} = r\mathbf{s}$, $r \geqslant 0$, $\mathbf{s} \in U_m = \{\mathbf{s} \in \mathbb{R}^m : \|\mathbf{s}\| = 1\}$, we have, with γ_m being the surface measure on U_m,

$$F_m((\mathbf{x}_1' \Sigma_{11}^{-1} \mathbf{x}_1)^{1/2}) = (\tfrac{1}{2}\det \Sigma_{11})^{-1/2} \int_0^\infty f(r^2)\, r^{m-1}\, dr \int_{U_m} \gamma_m(d\mathbf{s})\, e^{-r\mathbf{x}_1' B^{-1}\mathbf{s}}.$$

Putting $\mathbf{y}_1 = rB^{-1}\mathbf{x}_1$ we have $\|\mathbf{y}_1\|^2 = r^2 \mathbf{x}_1' B^{-1} \mathbf{x}_1 = 2r^2 \mathbf{x}_1' \Sigma_{11} \mathbf{x}_1$, and for $m = 1$

$$\int_{U_1} e^{-i\mathbf{y}_1 \mathbf{s}}\, \gamma_1(d\mathbf{s}) = \cos(\,|\,\mathbf{y}_1|),$$

and for $m \geqslant 2$

$$\begin{aligned}\int_{U_m} e^{-i\mathbf{y}_1'\mathbf{s}}\, \gamma_m(d\mathbf{s}) &= \int_0^\pi e^{-i\,\|\mathbf{y}_1\| \cos\theta} (\sin\theta)^{m-2}\, d\theta \\ &= \frac{\pi^{1/2}\Gamma((m-1)/2)}{(\|\mathbf{y}_1\|/2)^{(m-2)/2}} J_{(m-2)/2}(\|\mathbf{y}_1\|).\end{aligned}$$

where $J_\nu(\cdot)$ is the Bessel function of the first kind with $\nu > 0$.

It follows that

$$F_1(|x_1|\, \sigma_1^{-1}) = (\tfrac{1}{2}\sigma_1^2)^{-1/2} \int_0^\infty f(r^2) \cos(\sqrt{2}\, r\, |x_1|\, \sigma_1^{-1})\, dr$$

$$\begin{aligned}F_m((\mathbf{x}_1' \Sigma_{11}^{-1} \mathbf{x}_1)^{1/2}) &= \frac{(\tfrac{1}{2}\det \Sigma_{11})^{-1/2}\, \pi^{1/2}\Gamma((m-1)/2)}{(\tfrac{1}{2}\mathbf{x}_1 \Sigma_{11}^{-1} \mathbf{x}_1)^{(m-2)/4}} \\ &\quad \times \int_0^\infty r^{m/2} f(r^2)\, J_{(m-2)/2}(\sqrt{2}\, r(\mathbf{x}_1' \Sigma_{11}^{-1} \mathbf{x}_1)^{1/2})\, dr. \end{aligned} \tag{2.16}$$

The final expression for $S^2_{A,m}(x)$ now follows from (2.14)–(2.16).

It is clear from (2.1) that the conditional variance–covariance of $\mathbf{X}_2$ given $\mathbf{X}_1$ is proportional to its Gaussian counterpart, the constant conditional covariance matrix of $\mathbf{G}_2$ given $\mathbf{G}_1$, times a function $S^2_{A,m}(\cdot)$, depending on the dimensionality m of $\mathbf{X}_1$ and the distribution of A and evaluated at $(\mathbf{X}_1' \Sigma_{11}^{-1} \mathbf{X}_1)^{1/2}$. Thus, the heteroscedasticity of all conditional variances and covariances has a common functional form determined by the "conditional standard deviation factor" $S_{A,m}(x)$.

The expression in (2.2) is useful for evaluation when the distribution function of A is known explicitly. When this is not the case, but its Laplace transform is explicitly known, then the expressions in (2.4)–(2.5) are useful as illustrated below for the stable case.

Condition (2.3) can be expressed in terms of moments, by using

$$\int_0^\infty u^{p-1} E[A^k e^{-uA}]\,du = E\left[A^k \int_0^\infty u^{p-1} e^{-uA}\,du\right] = E[A^{k-p}]\,\Gamma(p).$$

Thus, condition (2.3) is equivalent to

$$\begin{aligned} &E[A^{-1/2}] < \infty \quad \text{and} \quad E[A^{1/2}] < \infty \quad \text{for} \quad m = 1, \\ \text{and} \qquad &E[A^{-m/2}] < \infty \quad \text{for} \quad m \geqslant 2. \end{aligned} \tag{2.17}$$

A useful alternative expression for $S^2_{A,m}(x)$ can be obtained in terms of the marginal density of the first component of the random vector $\mathbf{X}_1$ under the conditions in part (c) of Theorem 1.

COROLLARY 1. *Let $f_1(|x|/\sigma_{11})$ be the density of the first component of the random vector $\mathbf{X}$ (i.e., the density of $\mathbf{X}_1$ when $m = 1$) where σ^2_{11} is the $(1, 1)$ element of the covariance matrix Σ. Under the condition in Theorem* 1.III., *or* (2.17), *we have for $x > 0$,*

$$S^2_{A,1}(x) = \frac{\int_{x^2}^\infty f_1(u)\,du}{2f_1(x^2)} \tag{2.18.1}$$

$$S^2_{A,2k+1}(x) = -\frac{f_1^{(k-1)}(x^2)}{2f_1^{(k)}(x^2)}, \qquad k \geqslant 1 \tag{2.18.2}$$

$$S^2_{A,2}(x) = \frac{\int_0^\infty u^{1/2} f_1^{(1)}(x^2+u)\,du}{\int_0^\infty u^{-1/2} f_1^{(1)}(x^2+u)\,du} \tag{2.18.3}$$

$$S^2_{A,2k+2}(x) = -\frac{1}{2}\frac{\int_0^\infty u^{-1/2} f_1^{(k)}(x^2+u)\,du}{\int_0^\infty u^{-1/2} f_1^{(k+1)}(x^2+u)\,du}, \; k \geqslant 1. \tag{2.18.4}$$

Proof. It is known that (Kelker, 1970) since $\mathbf{X}_1$ is scale mixture of Normal distribution, i.e., has a spherical distribution, then the density $f_{\mathbf{X}_1}$ can be expressed as $f_{\mathbf{X}_1}(\mathbf{x}_1) = c_m g_m((\mathbf{x}_1' \Sigma_{11}^{-1} \mathbf{x}_1)^{1/2})$ for all $\mathbf{x}_1 \neq 0$, $m \geqslant 1$, where g_m is a function on $(0, \infty)$, and $c_m = (2\pi)^{-m/2}\,|\Sigma_{11}|^{-1/2}$. Clearly $(2\pi\sigma_{11})^{-1/2}\, g_1(|x|/\sigma_{11})$ is the density of the first component of $\mathbf{X}_1$. Since the integrand in (2.7) vanishes at 0, and A is assumed nondegenerate: $P(A = 0) < 1$, we have $0 < g_m(x) < \infty$ for all $x > 0$ and $m \geqslant 1$. Thus

$$S^2_{A,m}(x) = \frac{g_{m-2}(x)}{g_m(x)} \qquad x > 0, \quad m \geqslant 1. \tag{2.19}$$

Note that since (2.17) is satisfied, $g_{m-2}(x)$ is continuously differentiable over $x>0$ for $m \geqslant 1$, with

$$\frac{g'_{m-2}(x)}{g_m(x)} = \frac{-x \int_{[0,\infty)} u^{-m/2} \exp\left(-\frac{x^2}{2u}\right) dF_A(u)}{g_m(x)} = -x. \tag{2.20}$$

It follows from (2.19) and (2.20) that for $x>0$, $m \geqslant 1$,

$$\frac{[S^2_{A,m}(x)\, g_m(x)]'}{g_m(x)} = -x,$$

and thus $S^2_{A,m}(x)\, g_m(x) = \int_x^\infty u g_m(u)\, du$. Hence (2.19) can be expressed as

$$S^2_{A,m}(x) = \frac{\int_x^\infty u g_m(u)\, du}{g_m(x)}, \qquad x>0, \quad m \geqslant 1,$$

from which follows

$$S^2_{A,m}(x) = \frac{\int_{x^2}^\infty g_m(u^{1/2})\, du}{2 g_m(x)}. \tag{2.21}$$

We will now express all g_m's in terms of g_1. From the definition of g_m and (2.19), it follows

$$g_1^{(k)}(x^2) = \frac{(-1)^k}{2^k}\, g_{2k+1}(x^2)$$

$$\text{and} \qquad g_2^{(k)}(x^2) = \frac{(-1)^k}{2^k}\, g_{2k+2}(x^2), \quad k \geqslant 1,$$

and thus from (2.21)

$$S^2_{A,2k+1}(x) = -\frac{1}{2} \frac{g_1^{(k-1)}(x^2)}{g_1^{(k)}(x^2)}, \quad k \geqslant 1$$

$$\text{and} \qquad S^2_{A,2k+2}(x) = -\frac{1}{2} \frac{g_2^{(k-1)}(x^2)}{g_2^{(k)}(x^2)}, \quad k \geqslant 1. \tag{2.22}$$

It is easily checked that

$$g_2(x^2) = -\left(\frac{2}{\pi}\right)^{1/2} \int_0^\infty u^{-1/2} g_1^{(1)}(x^2+u)\, du, \tag{2.23}$$

from which it follows that

$$S^2_{A,\,2k+2}(x) = -\frac{1}{2}\,\frac{\int_0^\infty u^{-1/2} g_1^{(k)}(x^2+u)\,du}{\int_0^\infty u^{-1/2} g_1^{(k+1)}(x^2+u)\,du}, \qquad k \geqslant 1. \tag{2.24}$$

Thus (2.22) and (2.24) imply (2.18.2) and (2.18.4). These expressions of g_m, $m \geqslant 2$, in terms of g_1, in the more general setup of spherical distributions, are derived in Zolotarev, p. 286 (1981). Szablowski (1987) has obtained similar expressions for elliptically contoured measures. Also, (2.18.1) follows directly from (2.21) for $m=1$ and (2.18.3) follows from (2.21) and (2.23). Note that g is a functions of both m and the density of A. However, the subscript of A is omitted for easing the reading of the content, since this does not change for different values of m.

Corollary 1 ties with the methods of Zolotarev (1981) and Szablowski (1986, 1987). In their studies they evaluated elliptically contoured measures with respect to suitable chosen marginal densities or conditional variances and the distribution of $\mathbf{X}_1' \Sigma_{11}^{-1} \mathbf{X}_1$, which is the case here, where the conditional variance is expressed with respect to the first component of the vector $\mathbf{X}_1$.

We now consider in more detail the types of heteroscedasticity provided by this model by examining the universal standard deviation function $S_{A,m}(x)$. We first show that under assumptions even more restrictive than those in part (c) of Theorem 1, the value of $S_{A,m}(x)$ at $x=0$, as given by the expressions (2.2) or (2.4)–(2.5), exists and is finite, $S_{A,m}(x)$ is continuous, differentiable, and is approximately quadratic around zero.

Corollary 2. *If the equivalent assumptions* (2.3) *in Theorem* 1.III. *or* (2.17) *hold for* $m+2$, *then we have*

$$S_{A,m}(x) = S_{A,m}(0) + C_{A,m} x^2 + o(x^2) \qquad \text{as} \quad x \downarrow 0, \tag{2.25}$$

where $0 < S_{A,m}(0) < \infty$ *and* $S_{A,m}(0)$, $C_{A,m}$ *are given in terms of moments of* A: $\mu_{A,p} = E[A^p]\ -\infty < p < \infty$ *as follows*

$$S_{A,m}(0) = \frac{\mu^{1/2}_{A,\,-m/2+1}}{\mu^{1/2}_{A,\,-m/2}}, \qquad C_{A,m} = \frac{\mu_{A,\,-m/2+1}\mu_{A,\,-m/2-1} - \mu^2_{A,\,-m/2}}{4\mu^{3/2}_{A,\,-m/2}\mu^{1/2}_{A,\,-m/2+1}} \tag{2.26}$$

and in terms of the Laplace transform of A *by using*

$$\mu_{A,\,-k/2} = \frac{2M_k(L_A)}{\Gamma(k/2)}, \quad k = 1, 2, ..., m+2 \qquad \mu_{A,\,1/2} = \frac{2}{\sqrt{\pi}} M_1(-L_A'), \tag{2.27}$$

where $M_k(f) = \int_0^\infty r^{k-1} f(r^2)\,dr$.

Proof. From (2.19) we can write

$$
\begin{aligned}
S^2_{A,m}(x) - S^2_{A,m}(0) &= \frac{g_{m-2}(x)}{g_m(x)} - \frac{g_{m-2}(0)}{g_m(0)} \\
&= \frac{[g_{m-2}(x) - g_{m-2}(0)]\, g_m(0) - g_{m-2}(0)[g_m(x) - g_m(0)]}{g_m(x)\, g_m(0)} \\
&= \frac{1}{g_m(x)\, g_m(0)} \left\{ g_m(0) \int_0^x g'_{m-2}(y)\, dy - g_{m-2}(0) \int_0^x g'_m(y)\, dy \right\} \\
&= \frac{1}{g_m(x)\, g_m(0)} \left\{ -g_m(0) \int_0^x y g_m(y)\, dy + g_{m-2}(0) \int_0^x y g_{m+2}(y)\, dy \right\}
\end{aligned}
$$

and using

$$
\lim_{x \downarrow 0} \frac{1}{x^2} \int_0^x y g_m(y)\, dy = \lim_{x \downarrow 0} \frac{x g_m(x)}{2x} = \frac{1}{2}\, g_m(0)
$$

we obtain

$$
\lim_{x \downarrow 0} \frac{1}{x^2} \left[S^2_{A,m}(x) - S^2_{A,m}(0) \right] = \frac{1}{2 g^2_m(0)} \left\{ g_{m-2}(0)\, g_{m+2}(0) - g^2_m(0) \right\}.
$$

But the left-hand side is also

$$
\lim_{x \downarrow 0} \frac{2}{x^2} \left[S_{A,m}(x) - S_{A,m}(0) \right] S_{A,m}(0),
$$

and thus

$$
\lim_{x \downarrow 0} \frac{1}{x^2} \left[S_{A,m}(x) - S_{A,m}(0) \right] = \frac{g_{m-2}(0)\, g_{m+2}(0) - g^2_m(0)}{4 g^{3/2}_m(0)\, g^{1/2}_{m-2}(0)}.
$$

The expression in Corollary 2 follows by using $g_m(0) = E[A^{-m/2}] = \mu_{A,\,-m/2}$.

To express $S_{A,m}(0)$ and $C_{A,m}$ in terms of the Laplace transform of A, we could use the expressions (2.4)–(2.5) instead of (2.2) and follow a similar line of argument, or equivalently, we could express the moments $\mu_{A,\,p}$ in terms of the Laplace transform L_A. This is accomplished by evaluating

$$M_m(L_A) = E\left[\int_0^\infty r^{m-1} e^{-r^2 A}\, dr\right] = \frac{1}{2} E\left[\int_0^\infty x^{m/2-1} e^{-xA}\, dx\right]$$
$$= \frac{1}{2}\Gamma\left(\frac{m}{2}\right) E[A^{-m/2}]$$

for $m \geqslant 1$, so that $\mu_{A,\,-m/2} = 2M_m(L_A)/\Gamma(m/2)$, $m \geqslant 1$. This works for all the moments required in (2.25), with the exception of $\mu_{A,\,1/2}$ and $\mu_0 = 1$. This $\mu_{A,\,1/2}$, along with all $\mu_{A,\,-m/2}$, can be expressed using

$$M_m(-L'_A) = -\int_0^\infty r^{m-1} L'_A(r^2)\, dr = \int_0^\infty r^{m-1} E[Ae^{-r^2 A}]\, dr$$
$$= \frac{1}{2} E\left[A \int_0^\infty x^{m/2-1} e^{-xA}\, dx\right] = \frac{1}{2}\Gamma\left(\frac{m}{2}\right) E[A^{1-m/2}]$$
$$= \frac{1}{2}\Gamma\left(\frac{m}{2}\right) \mu_{A,\,1-m/2},$$

for $m = 1$, leading to $\mu_{A,\,1/2} = (2/\sqrt{\pi})\, M_1(-L'_A)$.

When the assumption in Corollary 2 is not satisfied, i.e., when $E[A^{-m/2+1}] = \infty$, a wide variety of (non-quadratic) asymptotic behavior at zero and at infinity is still possible. This results in a wide variety of heteroscedastic models illustrated in two examples in Section 3.

The following corollary describes the behavior of the factor $S_{A,\,m}(x)$ with respect to x for a given dimensionality. Furthermore, it shows how the higher the dimension we condition on, the lower the value of $S_{A,\,m}(x)$ becomes, for a given value of $x \in \mathbb{R}$.

Corollary 3. *If* $F_A(0) = 0$, *then* (i) *for any* $m \geqslant 1$, $S_{A,\,m}(x)$ *is non-decreasing in* $x > 0$, *and* (ii) *for any* $x \geqslant 0$, $S_{A,\,m}(x)$ *is non-increasing in* $m \geqslant 1$.

Proof. (i) Since

$$\frac{d}{dx} S^2_{A,\,m}(x) = \frac{d}{dx}\left(\frac{g_{m-2}(x)}{g_m(x)}\right) = \frac{g'_{m-2}(x)\, g_m(x) - g_{m-2}(x)\, g'_m(x)}{g^2_m(x)}$$
$$= \frac{x}{g^2_m(x)}\{g_{m-2}(x)\, g_{m+2}(x) - g^2_m(x)\}, \qquad x > 0.$$

It follows that $S_{A,m}(x)$ is nondecreasing if and only if the within the brackets quantity is greater or equal to zero, or

$$g_{m-2}(x)\, g_{m+2}(x) \geqslant g_m^2(x). \tag{2.28}$$

To show this, we proceed as follows. Let $A_x > 0$ denote the random variable associated with the random variable A, via the probability measure relationship

$$v_{A,x}(du) = \frac{u^{-(m+2)/2} e^{-x^2/2u}\, dF_A(u)}{\int_{[0,\infty)} u^{-(m+2)/2} e^{-x^2/2u}\, dF_A(u)}.$$

Assume $F_A(0) = 0$ to avoid some trivial difficulties.

Hence the necessary and sufficient condition (2.28) may be expressed in the form

$$\int_{[0,\infty)} u^2 v_{A,x}(du) \geqslant \left(\int_{[0,\infty)} u v_{A,x}(du) \right)^2, \qquad \text{or equivalently,} \qquad E[A_x^2] \geqslant E[A_x]^2,$$

which is always true.

(ii) To show that $S^2_{A,m}(x)$ is non-increasing with respect to $m = 1, 2, \ldots$ for fixed value of $x \in \mathbb{R}^+$, it is necessary and sufficient to show that $S^2_{A,m+1}(x) \leqslant S^2_{A,m}(x)$, $m = 1, 2, \ldots$, for fixed $x > 0$, or equivalently from (2.19), we need to show that $g_{m+1}(x)\, g_{m-2}(x) \geqslant g_m(x)\, g_{m-1}(x)$.

As in part (i), let $A_{x,1} > 0$ denote the random variable associated with the random variable A, and let $\theta_{A,x}(du)$ be modified version of $v_{A,x}(du)$ defined as follows:

$$\theta_{A,x}(du) = \frac{u^{-(m+1)/2} e^{-x^2/2u}\, dF_A(u)}{\int_{[0,\infty)} u^{-(m+1)/2} e^{-x^2/2u}\, dF_A(u)}.$$

Once again, the necessary and sufficient condition that the last inequality holds is to show that

$$\int_{[0,\infty)} u \theta_{A,x}(du) \int_{[0,\infty)} u^{1/2} \theta_{A,x}(du) \leqslant \int_{[0,\infty)} u^{3/2} \theta_{A,x}(du),$$

or equivalently,

$$E[A_{x,1}]\, E[A_{x,1}^{1/2}] \leqslant E[A_{x,1}^{3/2}]. \tag{2.29}$$

However, the last inequality is always true, since $E[A_x^p]^{1/p}$ is a non-decreasing function of $p>0$, and $E[A_{x,1}^{1/2}] \leqslant E[A_{x,1}^{3/2}]^{1/3}$, and $E[A_{x,1}] \leqslant E[A_{x,1}^{3/2}]^{2/3}$. Thus, by multiplying the last two inequalities, (2.29) is now evident. This completes the proof of Corollary 3.

3. EXAMPLES

In this section we analyze the behavior of $S_{A,m}(x)$ in two specific cases, (1) when the random variable A is uniform and (2) when it is a positive stable. All the proofs of the following results will be deferred to Section 4.

1. *Uniformly Distributed A.* Here A is uniformly distributed over $[a, b]$, $0 \leqslant a < b < \infty$.

First let $a>0$. Then, $E[A^p] < \infty$ for all $-\infty < p < \infty$, so by Corollary 1, all $S_{A,m}(x)$ are approximately quadratic around zero, i.e., (1.9) and (1.10) hold with

$$\mu_{A,p} = \frac{b^{p+1} - a^{p+1}}{(p+1)(b-a)} \qquad \text{for all} \quad p \in (-\infty, \infty) \qquad \text{except} \quad p = -1,$$

$$\mu_{A,-1} = \frac{1}{b-a} \ln\left(\frac{b}{a}\right).$$

It is not hard to see that

$$a^{1/2} \leqslant S_{A,m}(x) \leqslant b^{1/2}, \qquad \text{for all} \quad m = 1, 2, \ldots. \tag{3.1}$$

And at infinity all $S_m(x)$ tend to the same constant:

$$\lim_{x \to \infty} S_{A,m}(x) = b^{1/2}, \qquad \text{for all} \quad m = 1, 2, \ldots. \tag{3.2}$$

Specifically, it is shown that for sufficiently large x,

$$S_{A,m}(x) = \begin{cases} b^{1/2}\left(1 - \dfrac{b}{2x^2}\right) + o(x^{-2}) & \text{for} \quad m \neq 4 \\ b^{1/2}\left(1 - \dfrac{b}{x^2}\right) + o(x^{-2}) & \text{for} \quad m = 4. \end{cases} \tag{3.3}$$

Also from Corollary 3, $S_{A,m}(x)$, $m \geqslant 1$, increases from $S_{A,m}(0)$ to $b^{1/2}$.

Let $a=0$. Then, $E[A^p]<\infty$ only for $-1<p<\infty$ and thus $E[A^{-m/2-1}]=\infty$ for all $m\geqslant 1$, so Corollary 1 never applies. In this case, the limiting value of $S_{A,m}(x)$ at zero vanishes except when $m=1$:

$$\lim_{x\to 0} S_{A,1}(x)=\left(\frac{b}{3}\right)^{1/2} \quad \text{and} \quad \lim_{x\to 0} S_{A,m}(x)=0, \qquad m\geqslant 2. \tag{3.4}$$

The limiting value at infinity is as (2.25). Around zero, $S_{A,m}(x)$ is approximately linear for $m\geqslant 5$, whereas for smaller values of m it rises faster from its value at zero, the precise asymptotic expressions are presented as follows,

$$S_{A,1}(x)=\left(\frac{b}{3}\right)^{1/2}+o(x^2), \tag{3.5.1}$$

$$S_{A,2}(x)=b^{1/2}\left(\ln\frac{2b}{x^2}\right)^{-1/2}\left(1+\frac{\gamma}{2}\left(\ln\frac{2b}{x^2}\right)^{-1/2}\right)+o\left(\left(\ln\frac{1}{x}\right)^{3/2}\right), \tag{3.5.2}$$

$$S_{A,3}(x)=\left(\frac{b}{2\pi}\right)^{1/2} x\left(1+\frac{x}{\sqrt{2b}}\right)+o(x^2), \tag{3.5.3}$$

$$S_{A,4}(x)=\frac{x}{\sqrt{2}}\left(-\gamma+\ln\frac{2b}{x^2}\right)+o\left(x\ln\frac{1}{x}\right), \tag{3.5.4}$$

$$S_{A,m}(x)=\frac{x}{\sqrt{m-4}}\left(1-\frac{x^{m-1}}{2(2b)^{m/2-2}\,\Gamma\left(\frac{m}{2}-1\right)}\right)+o(x^m), \qquad m\geqslant 5, \tag{3.5.5}$$

where $\gamma=0.57721$, is the Euler's constant.

2. *Stable Distributed A*. Here $A\sim S_{\alpha/2}(\cos(\frac{\pi\alpha}{4}),1,0)$, $0<\alpha<2$, i.e., A is stable totally skewed to the right with $E[e^{-sA}]=e^{-s^{\alpha/2}}$. It will be shown that the scale factor, $S_{A,m}(x)$, which determines the shape of heteroscedasticity, can be expressed in an additive form with the dominant term being exactly the one we have achieved at infinity. On the other hand, the other term can be shown to explode to infinity with respect to x, except at $\alpha=1$, which is constant. This result supports Cioczek-Georges and Taqqu's (1993) arguments for $m=1$.

It can be shown that the scale factor associated with the variance-covariance matrix, $\mathrm{Cov}(\mathbf{X}_2\mid\mathbf{X}_1)$, $\mathbf{X}_1\in\mathbb{R}^m$, $m\geqslant 2$ has the following properties:

$$\lim_{x\to\infty}\frac{S^2_{A,m}(x)}{x^2}=\frac{1}{m+\alpha-2}. \tag{3.6}$$

The following result connects (3.6) by proving an additive relation, where the limiting term showing in (3.6) is one of the two terms.

$$S^2_{A,m}(x) = \frac{C(x;\alpha,m)}{4(m+\alpha-2)(m-1)} + \frac{x^2}{m+\alpha-2}, \tag{3.7}$$

where

$$C(x;\alpha,m) = \frac{\alpha^2(m-1)\int_{[0,\infty)} e^{-r^\alpha} r^{m/2+2(\alpha-1)} J_{(m-2)/2}(\sqrt{2}\,xr)\,dr}{\int_{[0,\infty)} e^{-r^\alpha} r^{m/2} J_{(m-2)/2}(\sqrt{2}\,xr)\,dr}$$

and $J_\nu(x)$ is the Bessel function of the first kind. It is also shown that

$$\lim_{x\to\infty}\left|S^2_{A,m}(x) - \frac{x^2}{m+\alpha-2}\right| = \begin{cases} \infty & \text{for} \quad \alpha \neq 1 \\ \dfrac{1}{4(m-1)} & \text{for} \quad \alpha = 1 \end{cases} \tag{3.8}$$

Remarks. When $\alpha=1$, the functional form of $S^2_{A,m}(x)$, for $m \geqslant 2$, becomes a pure quadratic function. This was also noticed by Cioczek-Georges and Taqqu (1993) for $m=1$ when they studied the behavior of their stable conditional variance. Therefore, for $\alpha=1$, the form is deduced to be

$$S^2_{A,m}(x) = \frac{1}{m-1}\left[x^2 + \frac{1}{4}\right], \qquad m \geqslant 2. \tag{3.9}$$

For completeness, we shall state the case $m=1$. This was approached by both Wu and Cambanis (1991) and Cioczek-Georges and Taqqu (1993) for the stable case. Here, it will be presented in the sub-Gaussian case. For $m=1$ the scale factor associated with the conditional variance, $Var(X_2 \mid X_1)$, has the properties

$$\begin{gathered} \lim_{x\to\infty} \frac{S^2_{A,1}(x)}{x^2} = \frac{1}{\alpha-1}, \\ S^2_{A,1}(x) = \frac{C(x;\alpha,1)}{2(\alpha-1)} + \frac{x^2}{\alpha-1}, \qquad \text{and} \qquad \lim_{x\to\infty}\left|S^2_{A,1}(x) - \frac{x^2}{\alpha-1}\right| = \infty, \end{gathered} \tag{3.10}$$

where

$$C(x;\alpha,1) = \frac{\alpha\int_{[0,\infty)} e^{-r^\alpha} r^{2(\alpha-1)} \cos(\sqrt{2}\,xr)\,dr}{\int_{[0,\infty)} e^{-r^\alpha} \cos(\sqrt{2}\,xr)\,dr}.$$

4. PROOFS OF SECONDARY RESULTS

In the proof of Theorem 1, we use the following form of the regular conditional distribution of A given X_1.

PROPOSITION 1. *For each non-negative measurable function $g(\cdot)$ we have*

$$E[g(A) \mid \mathbf{X}_1 = x_1] = \frac{\int_{[0,\infty)} g(u)\, u^{-m/2} \exp\left(-\frac{1}{2u} x_1' \Sigma_{11}^{-1} x_1\right) dF_A(u)}{\int_{[0,\infty)} u^{-m/2} \exp\left(-\frac{1}{2u} x_1' \Sigma_{11}^{-1} x_1\right) dF_A(u)}, \quad (4.1)$$

for almost every $x_1 \in \mathbb{R}^m$, where $F_A(\cdot)$ is the distribution function of A.

Proof. It is well known that the joint density function of $\mathbf{X}_1 \in \mathbb{R}^m$ with $\mathbf{X}_1 \overset{d}{=} A^{1/2}\mathbf{G}_1$, where $\mathbf{G}_1$ is a symmetric Gaussian random vector with covariance matrix Σ_{11} is of the form

$$f_{\mathbf{X}_1}(\mathbf{x}_1) = \frac{(\det \Sigma_{11})^{-1/2}}{(2\pi)^{m/2}} \int_{[0,\infty)} u^{-m/2} \exp\left(-\frac{1}{2u} \mathbf{x}_1' \Sigma_{11}^{-1} \mathbf{x}_1\right) dF_A(u). \quad (4.2)$$

The rest of the proof is a simple consequence of the conditional expectation and the formula of the joint distribution of $\mathbf{X}_1$ and A.

Proof of (3.1). The proof of this follows by just noting that

$$S^2_{A,m}(x) = \frac{\frac{x^2}{2} \int_{x^2/2b}^{x^2/2a} e^{-y} y^{m/2-3}\, dy}{\int_{x^2/2b}^{x^2/2a} e^{-y} y^{m/2-2}\, dy} \begin{cases} \leqslant b \dfrac{\int_{x^2/2b}^{x^2/2a} e^{-y} y^{m/2-2}\, dy}{\int_{x^2/2b}^{x^2/2a} e^{-y} y^{m/2-2}\, dy} = b \\[2ex] \geqslant a \dfrac{\int_{x^2/2b}^{x^2/2a} e^{-y} y^{m/2-2}\, dy}{\int_{x^2/2b}^{x^2/2a} e^{-y} y^{m/2-2}\, dy} = a. \end{cases}$$

Proof of (3.2), (3.3), and (3.4). It is known (see, e.g., Gradshteyn and Ryzhik, 1980, p. 943) that for sufficiently large values of x and for any $a \in \mathbb{R}$,

$$x^{-(a-1)} e^x \Gamma(a, x) = 1 - \frac{a-1}{x} + \frac{(a-1)(a-2)}{x^2} + o(x^{-2}),$$

where $\Gamma(a, x) = \int_x^\infty e^{-y} y^{a-1}\, dy$, is the incomplete gamma function. Hence, for any m except $m = 4$, 2 we have

$$
\begin{aligned}
&S^2_{A,m}(x)\\
&= \left[\frac{b\left(\frac{x^2}{2b}\right)^{-(m-6)/2} e^{x^2/2b}\,\Gamma\left(\frac{m-4}{2}, \frac{x^2}{2b}\right)}{\left(\frac{x^2}{2b}\right)^{(m-4)/2} e^{x^2/2b}} - \frac{a\left(\frac{x^2}{2a}\right)^{-(m-6)/2} e^{x^2/2b}\,\Gamma\left(\frac{m-4}{2}, \frac{x^2}{2b}\right)}{\left(\frac{x^2}{2a}\right)^{-(m-4)/2} e^{x^2/2b}} \right]\\
&\quad \times \left[\Gamma\left(\frac{m-4}{2}, \frac{x^2}{2b}\right) - \Gamma\left(\frac{m-4}{2}, \frac{x^2}{2a}\right) \right]^{-1}\\
&= \frac{b\left[1 + \frac{b(m-6)}{x^2} + o(x^{-2})\right]}{1 + \frac{b(m-4)}{x^2} + \left(\frac{a}{b}\right)^{-(m-4)/2} e^{-(x^2/2)(1/a - 1/b)} + o(1)}\\
&\quad - \frac{a\left[1 + \frac{a(m-6)}{x^2} + o(x^{-2})\right]}{\left(\frac{b}{a}\right)^{-(m-4)/2} e^{(x^2/2)(1/a - 1/b)} - 1 - \frac{a(m-4)}{x^2} + o(1)}\\
&= b\left(1 - \frac{b}{x^2}\right) + o(x^2).
\end{aligned}
$$

Taking the square root in both sides, the answer follows immediately.

Proof for $m = 4$. Call $\Gamma(0, x) = \int_x^\infty (e^u/u)\, du$. Hence, via Lemma 1

$$
\begin{aligned}
S^2_{A,4}(x) &= \frac{b\,\frac{x^2}{2b}\, e^{x^2/2a}\,\Gamma\left(0, \frac{x^2}{2b}\right)}{e^{x^2/2a}\left[\Gamma\left(1, \frac{x^2}{2b}\right) - \Gamma\left(1, \frac{x^2}{2a}\right)\right]} + o(x^{-2})\\
&= \frac{b\left(1 - \frac{2b}{x^2}\right)}{1 - e^{-(x^2/2)(1/a - 1/b)}} + o(x^{-2}) = b\left(1 - \frac{2b}{x^2}\right) + o(x^{-2}).
\end{aligned}
$$

This completes the proof for $m = 4$.

Proof for $m=2$. In exactly the same fashion as above, we note that

$$S^2_{A,2}(x)=\frac{\dfrac{x^2}{2}\left[\Gamma\left(-1,\dfrac{x^2}{2b}\right)-\Gamma\left(-1,\dfrac{x^2}{2a}\right)\right]}{\Gamma\left(0,\dfrac{x^2}{2b}\right)-\Gamma\left(0,\dfrac{x^2}{2a}\right)}$$

$$=\frac{b(1+\dfrac{b}{x^2}+o(x^{-2})\Big)}{1+\dfrac{2b^2}{x^2}+o(x^{-2})}=b\left(1-\frac{b}{x^2}\right)+o(x^{-2}).$$

Proof of (3.5.1). If $m=1$, it can be seen that

$$S^2_{A,1}(x)=\frac{\dfrac{x^2}{2}\displaystyle\int_{x^2/2b}^{\infty}\frac{e^{-y}}{y^{5/2}}\,dy}{\displaystyle\int_{x^2/2b}^{\infty}\frac{e^{-y}}{y^{3/2}}\,dy}.$$

Now using integration by parts we have that

$$\int_x^{\infty}\frac{e^{-y}}{y^{a+1}}\,dy=\frac{1}{a}\left[\frac{e^{-x}}{x^a}-\int_x^{\infty}e^{-y}y^{-a}\,dy\right]$$

and Lemma 2, and it follows that the expression above may be written

$$S^2_{A,1}(x)=\frac{2b}{3}\left\{\frac{1}{\left(\dfrac{x^2}{2b}\right)^{1/2}e^{x^2/2b}\displaystyle\int_{x^2/2b}^{\infty}\frac{e^{-y}}{y^{3/2}}\,dy}-\frac{x^2}{2b}\right\}$$

$$=\frac{b}{3}\left\{\frac{1}{1-\dfrac{x^2}{b}+o(x^2)}-\frac{x^2}{b}\right\}=\frac{b}{3}\{1+o(x^2)\}.$$

Proof of (3.5.2). Repeating the same arguments, we may also have that

$$S^2_{A,2}(x)=\frac{\dfrac{x^2}{2}\displaystyle\int_{x^2/2b}^{\infty}y^{-2}e^{-y}\,dy}{\displaystyle\int_{x^2/2b}^{\infty}y^{-1}e^{-y}\,dy}=b\left\{\frac{1}{e^{x^2/2b}\displaystyle\int_{x^2/2b}^{\infty}e^{-y}y^{-1}\,dy}+\frac{x^2}{b}\right\}.$$

Thus, since

$$\int_x^{\infty}\frac{e^{-u}}{u}\,du=-\gamma+\ln\frac{1}{x}+x+o(x),\ x>0, \tag{4.3}$$

(Hardy, 1949, p. 27), and $e^x = 1 + x + o(x)$, it implies that

$$S^2_{A,2}(x) = \frac{b}{\left(1 + \frac{x^2}{2b} + (x^2)\right) \ln(2b/x^2) \left(1 - \frac{\gamma}{\ln(2b/x^2)} + o\left(\frac{1}{\ln(2b/x^2)}\right)\right)} + \frac{x^2}{2}$$

$$= \frac{b}{\ln(2b/x^2)} \left\{1 + \frac{\gamma}{\ln(2b/x^2)} + o\left(\frac{1}{\ln(2b/x^2)}\right)\right\},$$

where $\gamma = .57721$, is the Euler's constant.

Proof of (3.5.3). Since

$$\Gamma(a, x) = \Gamma(a) - \sum_{n=0}^{\infty} \frac{(-1)^n x^{a+n}}{n!(a+n)}, \qquad \text{for} \quad a > 0 \tag{4.4}$$

(Gradshteyn and Ryzhik, 1980, p. 941) and since $e^x = 1 + x + o(x)$ around the origin, for small argument of x we obtain that

$$S^2_{A,3}(x) = b \left\{ \frac{\left(\frac{x^2}{2b}\right)^{1/2} e^{-x^2/2b}}{\int_{x^2/2b}^{\infty} e^{-y} y^{-1/2}\, dy} - \frac{x^2}{b} \right\}$$

$$= b^{1/2} \frac{x}{\sqrt{2}} \frac{1}{\Gamma(1/2)} \left\{1 + 2\frac{x}{\sqrt{2b}} + o(x)\right\}.$$

Proof of (3.5.4). In connection with Lemma 3, it follows that

$$S^2_{A,4}(x) = \frac{x^2/2b \int_{x^2/2b}^{\infty} e^{-y} y^{-1}\, dy}{\int_{x^2/2b}^{\infty} e^{-y}\, dy} = \frac{x^2}{2} \left\{ -\gamma + \ln \frac{2b}{x^2} \right\} + o\left(x^2 \ln \frac{1}{x}\right).$$

Proof of (3.5.5). For $m \geqslant 5$, we just utilize (4.4),

$$S^2_{A,m}(x) = \frac{x^2/2b \int_{x^2/2b}^{\infty} e^{-y} y^{(m-6)/2}\, dy}{\int_{x^2/2b}^{\infty} e^{-y} y^{(m-4)/2}\, dy}$$

$$= \frac{x^2}{2} \frac{\Gamma\left(\frac{m-4}{2}\right)}{\Gamma\left(\frac{m-2}{2}\right)} \frac{1 - \frac{(x^2/2b)^{(m-4)/2}}{\Gamma((m-2)/2)} + o(x^{m-1})}{1 - \frac{(x^2/2b)^{(m-2)/2}}{\Gamma(m/2)} + o(x^m)}$$

$$= \frac{x^2}{m-4} \left\{1 - \frac{x^{m-1}}{(2b)^{m/2-2}\, \Gamma(m/2-1)} + o(x^{m-1})\right\}.$$

This completes the proof of (3.5.5).

In establishing Theorem 2, we are aided by using some ideas from the Tauberian Theorem (see, e.g., Bingham *et al.*, 1987). We incorporate relations and identities given in Cambanis and Fotopoulos (1995), and we utilize various properties of the Bessel family. We continue by first resolving Theorem 1 and then Lemma 3.

Proof of (3.6). Since the choice of A is such that $A \sim S_{\alpha/2}(\sigma, 1, 0)$, $0 < \alpha < 2$, $\sigma > 0$, we have that

$$P(A > x) \sim \frac{\sigma^{\alpha/2}}{\Gamma(1-\alpha/2)\cos \pi\alpha/4}\, x^{-\alpha/2} = c_{\sigma,\alpha} x^{-\alpha/2} \qquad \text{as} \quad x \to \infty. \tag{4.5}$$

At this point we are interested to know the behavior of $g_m(x)$ as $x \to \infty$ occurred in (3.14) with the scalar being stable, and consequently to determine the behavior of $S^2_{A,m}(x)$ for large arguments of x. We shall cover both cases $m \geqslant 2$ with $\alpha \in (0, 2)$, and $m = 1$ with $\alpha \in (1, 2)$. Using integration by parts, it follows that

$$\begin{aligned} g_m(x) &= -\int_{[0,\infty} u^{-m/2} e^{-x^2/2u}\, dP(A > u) \qquad (4.6) \\ &= -u^{-m/2} e^{-x^2/2u} P(A > u)\big|_0^\infty + \int_{[0,\infty)} P(A > u)\, d[u^{-m/2} e^{-x^2/2u}] \\ &= \int_{[0,\infty)} e^{-x^2/2u} \left[-\frac{m}{2} u^{-(m+2)/2} + \frac{x^2}{2} u^{-(m+4)/2} \right] P(A > u)\, du \\ &= \left(\frac{x^2}{2}\right)^{-m/2} \int_{[0,\infty)} e^{-x^2/2u} \left[-\frac{m}{2} \left(\frac{x^2}{2u}\right)^{(m+2)/2} \right] P\left(A > \frac{2u}{x^2}\frac{x^2}{2}\right) d\left(\frac{2u}{x^2}\right) \\ &= \left(\frac{x^2}{2}\right)^{-(m-2)/2} \int_{[0,\infty)} e^{-y} \left[-\frac{m}{2} y^{(m-2)/2} + y^{m/2} \right] P\left(A > \frac{x^2}{2y}\right) dy. \end{aligned}$$

In connection (3.27), it follows that for $m \geqslant 2$, $\alpha \in (0, 2)$ and $m = 1$, $\alpha \in (1, 2)$ and for $x \to \infty$,

$$\begin{aligned} g_m(x) &\sim c_{\sigma,\alpha} \left(\frac{x^2}{2}\right)^{(m-2+\alpha)/2} \int_{[0,\infty)} e^{-y} \left\{ -\frac{m}{2} y^{(m+\alpha-2)/2} \right\} dy \qquad (4.7) \\ &= c_{\sigma,\alpha} \left(\frac{x^2}{2}\right)^{(m-2+\alpha)/2} \Gamma\left(\frac{m+\alpha}{2}\right) \frac{\alpha}{2}, \end{aligned}$$

which leads to

$$S_{A,m}^2(x)=\frac{g_{m-2}(x)}{g_m(x)}\sim\frac{c_{\sigma,\alpha}\frac{\alpha}{2}\left(\frac{x^2}{2}\right)^{-(m-4+\alpha)/2}\Gamma\left(\frac{m-2+\alpha}{2}\right)}{c_{\sigma,\alpha}\frac{\alpha}{2}\left(\frac{x^2}{2}\right)^{-(m-2+\alpha)/2}\Gamma\left(\frac{m+\alpha}{2}\right)}=\frac{x^2}{m+\alpha-2} \tag{4.8}$$

This completes the proof of part (3.6).

Remark. Obviously, if $m=1$ and $\alpha\in(0,1)$, then $E[A^{1/2}]=\int_{[0,\infty)}u^{1/2}\exp(-\frac{x^2}{2u})\,dF_A(u)=\infty$. This follows from the fact that $u^{1/2}P(A>u)\uparrow\infty$ as $u\to\infty$, this is true, because $u^{1/2}P(A>u)\sim c_{\sigma,\alpha}u^{(1-\alpha)/2}\to\infty$, as $u\to\infty$ for $\alpha\in(0,1)$. This concludes that $E[A^{1/2}]=\infty$.

Proof of (3.7). For simplicity, we set

$$\alpha=\left(\frac{\sigma}{\cos\frac{\pi\alpha}{4}}\right)^{2/\alpha}=1.$$

Call

$$A(x;\alpha,m)=\frac{\int_{[0,\infty)}e^{-r^\alpha}r^{(m+\alpha)/2}J_{(m-2)/2}(\sqrt{2}\,xr)\,dr}{\int_{[0,\infty)}e^{-r^\alpha}r^{m/2}J_{(m-2)/2}(\sqrt{2}\,xr)\,dr}, \tag{4.9}$$

$$B(x;\alpha,m)=\alpha(m+\alpha-2)\frac{m}{2}\sqrt{2}\,x\frac{\int_{[0,\infty)}e^{-r^\alpha}r^{(m+\alpha-4)/2}J_{(m-2)/2}(\sqrt{2}\,xr)\,dr}{\int_{[0,\infty)}e^{-r^\alpha}r^{m/2}J_{(m-2)/2}(\sqrt{2}\,xr)\,dr} \tag{4.10}$$

$$=(m+\alpha-2)\,m\sqrt{2}\,xS_{A,m}^2(x), \tag{4.11}$$

and

$$C(x;\alpha,m)=\alpha^2\frac{m}{2}\sqrt{2}\,x\frac{\int_{[0,\infty)}e^{-r^\alpha}r^{(m+\alpha-4)/2}J_{(m-2)/2}(\sqrt{2}\,xr)\,dr}{\int_{[0,\infty)}e^{-r^\alpha}r^{m/2}J_{(m-2)/2}(\sqrt{2}\,xr)\,dr}. \tag{4.12}$$

Hence, from Lemma 3, we obtain that

$$\begin{aligned}(m+\alpha-2)\,m\sqrt{2}\,xS_{A,m}^2(x)&=-A(x;\alpha,m)+B(x;\alpha,m)+\frac{m}{2}(\sqrt{2}\,x)^3\\&=C(x;\alpha,m)-B(x;\alpha,m)+B(x;\alpha,m)\\&\quad+\frac{m}{2}(\sqrt{2}\,x)^3=C(x;\alpha,m)+2(m-1)\,x^2.\end{aligned} \tag{4.13}$$

This completes the proof of part (3.7).

Proof of (3.8). For convenience, set $\lambda = \sqrt{2}\, x$. From (4.13), it follows that

$$\frac{m}{2\lambda} C(x; \alpha, m) = \alpha^2 \frac{\int_{[0,\infty)} e^{-r^\alpha} r^{(m+4\alpha-4)/2} J_{(m-2)/2}(\lambda r)\, dr}{\int_{[0,\infty)} e^{-r^\alpha} r^{m/2} J_{(m-2)/2}(\lambda r)\, dr}$$

$$= \frac{\alpha^2}{\lambda^{2(\alpha-1)}} \frac{\int_{[0,\infty)} e^{-(u/\lambda)^\alpha} u^{(m+2\alpha-4)/2} J_{(m-2)/2}(u)\, du}{\int_{[0,\infty)} e^{-(u/\lambda)^\alpha} u^{m/2} J_{(m-2)/2}(u)\, du}$$

$$= \frac{\alpha^2}{\lambda^{2(\alpha-1)}} \frac{N(x; \alpha, m)}{D(x; \alpha, m)}. \tag{4.14}$$

We first examine $N(x; \alpha, m)$. It is clear that

$$N(x; \alpha, m) = \int_{[0,\Delta)} + \int_{[\Delta,\infty)} = I_1 + I_2, \qquad \text{for} \quad \Delta = \Delta(\lambda). \tag{4.15}$$

We take $\Delta/\lambda < 1$, $\Delta/\lambda \to 0$, as $\lambda \uparrow \infty$ and both Δ and λ tend to infinity. It can be checked that

$$\begin{aligned} I_1 = {} & \frac{1}{\lambda} \int_{[0,\Delta)} \frac{e^{-(u/\lambda)^\alpha} - 1}{1/\lambda} u^{(m+4(\alpha-1))/2} J_{(m-2)/2}(u)\, du \\ & + \int_{[\Delta,\infty)} u^{(m+4(\alpha-1))/2} J_{(m-2)/2}(u)\, du \\ \sim {} & -\frac{1}{\lambda^\alpha} \int_{[0,\Delta)} u^{(m+6\alpha-4)/2} J_{(m-2)/2}(u)\, du + \int_{[\Delta,\infty)} u^{(m+4(\alpha-1))/2} J_{(m-2)/2}(u)\, du, \end{aligned} \tag{4.16}$$

since as $x \downarrow 0$, $\frac{e^{-x^\alpha}-1}{x} = x^{\alpha-1} + O(x^{2\alpha-1})$.

Obviously, the members on the right-hand side of (4.16) are in form of Lemma 5. From Lemmas 7 and 8 it can be seen that the dominant contribution of the right hand side of Lemma 6 is emanating from "$aJ_{\nu-1}(a)\, S_{\mu,\nu}(a)$".

From Lemma 6 and 8, we have that as $x \to \infty$,

$$J_\nu(x) = \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{2\nu+1}{4}\pi\right) + o(x^{-1/2})$$

$$\text{and} \qquad S_{\mu,\nu}(x) = x^{\mu-1} + O(x^{\mu-2}) \qquad \text{for} \quad p = 1. \tag{4.17}$$

In conjunction with Lemma 5 and (4.16), (4.17) becomes

$$I_1 \sim \sqrt{\frac{2}{\pi}} \cos\left(\Delta - \frac{m-1}{4}\pi\right) \left[\frac{\Delta^{(m-6\alpha-5)/2}}{\lambda^\alpha} - \Delta^{(m-6\alpha-5)/2}\right]. \tag{4.18}$$

Next, we consider I_2. By Lemma 6

$$\begin{aligned}
I_2 &= \int_{[\Delta,\infty)} e^{-(u/\lambda)^\alpha} u^{2(\alpha-1)} u^{m/2} J_{(m-2)/2}(u)\,du \\
&= \int_{[\Delta,\infty)} e^{-(u/\lambda)^\alpha} u^{2(\alpha-1)}\, d\int_{[0,u)} y^{m/2} J_{(m-2)/2}(y)\,dy \\
&= -e^{-(\Delta/\lambda)^\alpha}\,\Delta^{2(\alpha-1)+m/2} J_{m/2}(\Delta) - \int_{[\Delta,\infty)} u^{m/2} J_{m/2}(u)\, d[e^{-(u/\lambda)^\alpha} u^{2(\alpha-1)}] \\
&= -e^{-(\Delta/\lambda)^\alpha}\,\Delta^{2(\alpha-1)+m/2} J_{m/2}(\Delta) + \frac{\alpha}{\lambda^\alpha}\int_{[\Delta,\infty)} e^{-(u/\lambda)^\alpha} u^{3(\alpha-1)+m/2} J_{m/2}(u)\,du \\
&= -2(\alpha-1)\int_{[\Delta,\infty)} e^{-(u/\lambda)^\alpha} u^{2\alpha-3+m/2} J_{m/2}(u)\,du \\
&= I_{21} + \alpha I_{22} - 2(\alpha-1)\, I_{23}, \qquad \text{say.}
\end{aligned} \tag{4.19}$$

In view of (4.16) and (4.17), we obtain

$$I_{21} \sim \sqrt{\frac{2}{\pi}} \cos\left(\Delta - \frac{m+1}{4}\pi\right) \Delta^{(m+4\alpha-5)/2}. \tag{4.20}$$

To obtain I_{22}, some additional algebra is needed. From (4.16)

$$\begin{aligned}
I_{22} &= \sqrt{\frac{2}{\pi}}\,\lambda^{-\alpha}\int_{[\Delta,\infty)} e^{-(u/\lambda)^\alpha} u^{(m-6\alpha-7)/2}\,du \\
&\sim \frac{1}{\alpha}\sqrt{\frac{2}{\pi}}\,\frac{\lambda^{(m-6\alpha-5)/2}}{\lambda^\alpha}\int_{[(\Delta/\lambda)^\alpha,\infty)} y^{(m-4\alpha-5)/2\alpha} e^{-y}\,dy \\
&\sim \frac{1}{\alpha}\sqrt{\frac{2}{\pi}}\,\frac{\lambda^{(m+6\alpha-5)/2}}{\lambda^\alpha}\, e^{-(\Delta/\lambda)^\alpha}\left(\frac{\Delta}{\lambda}\right)^{(m+4\alpha-5)/2} \sim \frac{1}{\alpha}\sqrt{\frac{2}{\pi}}\,\Delta^{(m+4\alpha-5)/2}.
\end{aligned} \tag{4.21}$$

In exactly the same way we continue for I_{23},

$$I_{23} \sim \frac{1}{\alpha}\sqrt{\frac{2}{\pi}}\,\lambda^{(m+4\alpha-5)/2}\int_{[(\Delta/\lambda)^\alpha,\infty)} y^{(m+2\alpha-5)/2\alpha} e^{-y}\,dy \sim \frac{1}{\alpha}\sqrt{\frac{2}{\pi}}\,\lambda^\alpha\,\Delta^{(m+2\alpha-5)/2}. \tag{4.22}$$

Combining (4.16), (4.18)–(4.22), (4.15) becomes

$$|N(\lambda;\alpha,m)| = \sqrt{\frac{2}{\pi}}\,\Delta^{(m-1)/2}\left[c_1\,\frac{\Delta^{3\alpha-2}}{\lambda^\alpha} + c_2\,\Delta^{2(\alpha-1)} + c_3\lambda^\alpha\,\Delta^{\alpha-2}\right], \tag{4.23}$$

where c_1, c_2 and c_3 are positive suitable constants.

We proceed by investigating the behavior of the denominator.

$$D(\lambda;\alpha,m)=\int_{[0,\Delta)}+\int_{[\Delta,\infty)}=I'_1+I'_2,\qquad\text{say.}\tag{4.24}$$

Using identical arguments as before and Lemma 6, we have that

$$\begin{aligned}I'_1&=\frac{1}{\lambda^\alpha}\int_{[0,\Delta)}\frac{e^{-(u/\lambda)^\alpha}-1}{1/\lambda}u^{m/2}J_{(m-2)/2}(u)\,du+\int_{[0,\Delta)}u^{m/2}J_{(m-2)/2}(u)\,du\\&\sim\sqrt{\frac{2}{\pi}}\frac{1}{\lambda^\alpha}\cos\left(\Delta-\frac{m-1}{4}\pi\right)\Delta^{(m+2\alpha-1)/2}+\sqrt{\frac{2}{\pi}}\cos\left(\Delta-\frac{m+1}{4}\pi\right)\Delta^{(m-1)/2}.\end{aligned}\tag{4.25}$$

Applying similar ideas as in (4.19), it follows that

$$\begin{aligned}I'_2&=-e^{-(\Delta/\lambda)^\alpha}\Delta^{m/2}J_{m/2}(\Delta)+\frac{\alpha}{\lambda^\alpha}\int_{[\Delta,\infty)}e^{-(u/\lambda)^\alpha}u^{(m+2\alpha-2)/2}J_{m/2}(u)\,du\\&\sim-I'_{21}+\alpha I'_{22},\qquad\text{say.}\end{aligned}\tag{4.26}$$

Clearly,

$$I'_{21}\sim\sqrt{\frac{2}{\pi}}\cos\left(\Delta-\frac{m+1}{4}\pi\right)\Delta^{(m-1)/2}\tag{4.27}$$

and

$$I'_{22}\sim\frac{1}{\lambda^\alpha}\int_{[\Delta,\infty)}e^{-(u/\lambda)^\alpha}u^{(m+2\alpha-3)/2}\,du\sim\sqrt{\frac{2}{\pi}}\cos\left(\Delta-\frac{m+1}{4}\pi\right)\Delta^{(m-1)/2}.\tag{4.28}$$

Combining (4.25)–(4.28), (4.24) becomes

$$D(\lambda;\alpha,m)\sim\sqrt{\frac{2}{\pi}}\Delta^{(m-1)/2}\left[\frac{1}{\lambda^\alpha}\cos\left(\Delta-\frac{m-1}{4}\pi\right)\Delta^\alpha+\alpha\right].\tag{4.29}$$

In connection with (3.44) and (3.50), (3.35) becomes

$$\frac{m}{2\lambda}C(x;\alpha,m)\sim C_1\left(\frac{\Delta}{\lambda}\right)^{2(\alpha-1)}+c_2\left(\frac{\Delta}{\lambda}\right)^{\alpha-2}+c_3\left(\frac{\Delta}{\lambda}\right)^2,\tag{4.30}$$

where c_1, c_2, and c_3 are positive constants. This completes the proof of (3.8).

5. AUXILIARY RESULTS

LEMMA 1. *For sufficiently large x,*

$$e^x \int_x^\infty \frac{e^{-u}}{u}\,du = \frac{1}{x} \sum_{j=0}^m (-1)^j \frac{j!}{x^j} + o(x^{-m}).$$

Proof. Note that

$$\begin{aligned} e^x \int_x^\infty \frac{e^{-u}}{u}\,du &= \int_0^\infty \frac{e^{-v}}{v+x}\,dv = \frac{1}{x}\int_0^\infty \frac{e^{-v}}{1+v/x}\,dv = \frac{1}{x}\left(\int_0^x + \int_x^\infty\right) \\ &= \frac{1}{x}\sum_{j=0}^m (-1)^j \frac{1}{x^j}\int_0^x e^{-v} v^j\,dv + o(x^{-m}) + O(e^{-x}) \\ &= \frac{1}{x}\sum_{j=0}^m (-1)^j \frac{j!}{x^j} + o(x^{-m}). \end{aligned}$$

This completes the proof of the lemma.

LEMMA 2. *For $a < 1$,*

$$x^a e^x \int_x^\infty \frac{e^{-y}}{y^{a+1}}\,dy = \frac{1}{a} - \frac{x}{a(1-a)} + o(x^2) \qquad \text{as} \quad x \downarrow 0.$$

Proof. This is an outcome of a simple integration by parts arguments.

LEMMA 3. *For any $k = 0, 1, 2, \ldots$ the following recurrent relations are true*:

$$\text{(i)} \quad \left(\frac{1}{r}\frac{d}{dr}\right)^k (r^\nu J_\nu(r)) = r^{\nu-k} J_{\nu-k}(r)$$

and

$$\text{(ii)} \quad \left(\frac{1}{r}\frac{d}{dr}\right)^k (r^{-\nu} J_\nu(r)) = (-1)^k\, r^{-(\nu+k)} J_{\nu+k}(r).$$

LEMMA 4. *Let* $I(\lambda; m, a) = \int_{[0,\infty)} e^{-r^a} r^{m+a-1} \int_{[0,\infty)} \cos(\lambda r \cos\theta) \times \sin^m\theta\, d\theta dr$. *Then*

(i) $$a \frac{\left(\frac{\lambda}{2}\right)^{m/2}}{\sqrt{\pi}\,\Gamma\left(\frac{m+1}{2}\right)} I(\lambda; m, a) = \lambda \int_{[0,\infty)} e^{-r^a} r^{m/2} J_{(m-2)/2}(\lambda r)\, dr$$

(ii) $$\lambda \frac{\left(\frac{\lambda}{2}\right)^{m/2}}{\sqrt{\pi}\,\Gamma\left(\frac{m+1}{2}\right)} I(\lambda; m, a)$$

$$= (m+a-2) \int_{[0,\infty)} e^{-r^a} r^{(m+2a-4)/2} J_{(m-2)/2}(\lambda r)\, dr$$

$$- a \int_{[0,\infty)} e^{-r^a} r^{(m+4(a-1))/2} J_{(m-2)/2}(\lambda r)\, dr.$$

Proof. (i) Via Lemma 2(i), and a simple integration by parts, we proceed as follows:

$$\begin{aligned} a \frac{\left(\frac{\lambda}{2}\right)^{m/2}}{\sqrt{\pi}\,\Gamma\left(\frac{m+1}{2}\right)} I(\lambda; a, m) &= a \int_{[0,\infty)} e^{-r^a} r^{(m+2a-2)/2} J_{m/2}(\lambda r)\, dr \\ &= -\int_{[0,\infty)} r^{m/2} J_{m/2}(\lambda r)\, de^{-r^a} \\ &= \int_{[0,\infty)} e^{-r^a} r \left(\frac{1}{r}\, dr^{m/2} J_{m/2}(\lambda r)\right) \\ &= \lambda \int_{[0,\infty)} e^{-r^a} r^{m/2} J_{m/2}(\lambda r)\, dr. \end{aligned} \tag{5.1}$$

This completes the proof of (i).

(ii) using Lemma 2(ii), we have that

$$\lambda \frac{\left(\frac{\lambda}{2}\right)^{m/2}}{\sqrt{\pi}\,\Gamma\left(\frac{(m+1)}{2}\right)} I(\lambda; a, m)$$

$$= \lambda \int_{[0,\infty)} e^{-r^a} r^{m+a-1} \lambda^{m/2} ((\lambda r)^{-m/2} J_{m/2}(\lambda r))\, dr$$

$$
\begin{aligned}
&= -\int_{[0,\infty)} e^{-r^a} r^{m+a-2} \lambda^{m/2} \frac{d}{d\lambda r}((\lambda r)^{-(m-2)/2} J_{(m-2)/2}(\lambda r))\, dr \\
&= \frac{m-2}{2} \int_{[0,\infty)} e^{-r^a} r^{m+a-2} \lambda^{m/2} (\lambda r)^{-m/2} J_{(m-2)/2}(\lambda r)\, dr \\
&\quad - \int_{[0,\infty)} e^{-r^a} r^{m+a-2} \lambda^{m/2} \frac{(\lambda r)^{-(m-2)/2}}{\lambda}\, dJ_{(m-2)/2}(\lambda r) \\
&= \frac{m-2}{2} \int_{[0,\infty)} e^{-r^a} r^{(m+2a-4)/2} J_{(m-2)/2}(\lambda r)\, dr \\
&\quad - \int_{[0,\infty)} e^{-r^a} r^{(m+2a-2)/2}\, dJ_{(m-2)/2}(\lambda r) \\
&= \frac{m-2}{2} \int_{[0,\infty)} e^{-r^a} r^{(m+2a-4)/2} J_{(m-2)/2}(\lambda r)\, dr \\
&\quad + \int_{[0,\infty)} J_{(m-2)/2}(\lambda r)\, d\{e^{-r^a} r^{(m+2a-2)/2}\} \\
&= \frac{m-2}{2} \int_{[0,\infty)} e^{-r^a} r^{(m+2a-4)/2} J_{(m-2)/2}(\lambda r)\, dr \\
&\quad - a \int_{[0,\infty)} e^{-r^a} r^{(m+4(a-1))/2} J_{(m-2)/2}(\lambda r)\, dr \\
&\quad + \left(\frac{m}{2} + a - 1\right) A_{m/2}(\lambda) \int_{[0,\infty)} e^{-r^a} r^{(m/2+a-2)} J_{(m-2)/2}(\lambda r)\, dr \\
&= (m+a-2) \int_{[0,\infty)} e^{-r^a} r^{(m+2a-2)/2} J_{(m-2)/2}(\lambda r)\, dr \\
&\quad - a \int_{[0,\infty)} e^{-r^a} r^{(m+4(a-1))/2} J_{(m-2)/2}(\lambda r)\, dr.
\end{aligned}
$$

This completes the proof of Lemma 3.

LEMMA 5 (Gradsteyn and Ryzhik, 1980, p. 684, Eq. 6.56.13). *For $a > 0$ and $\mu + \nu > 0$,*

$$
\begin{aligned}
a^{\mu+1} \int_{[0,1)} x^\mu J_\nu(ax)\, dx = \int_{[0,\infty)} x^\mu J_\nu(x)\, dx \\
&= (\nu + \mu - 1)\, a J_\nu(a) + S_{\mu-1,\nu-1}(a) - a J_{\nu-1}(a)\, S_{\mu,\nu}(a) \\
&\quad + 2^\mu \frac{\Gamma((1+\mu+\nu)/2)}{\Gamma((1+\nu-\mu)/2)}
\end{aligned}
$$

is always true, where $S_{\mu,\nu}(x)$ is Lommel's function.

LEMMA 6 (Gradsteyn and Ryzhik, 1980, p. 683, Eq. 6.56.5). *For $v > 0$, the following equality holds*:

$$a^v \int_{[0,1)} x^v J_{v-1}(a, x)\, dx = \int_{[0,a)} x^v J_{v-1}(x)\, dx = a^v J_v(a).$$

LEMMA 7 (Abramowitz and Stegun, 1972, p. 364). *When v is fixed and $x \to \infty$*,

$$J_v(x) = \sqrt{\frac{2}{\pi x}} \{P(v, x) \cos \chi - \chi(v, x) \sin \chi\},$$

where

$$\chi = x - \left(\frac{1}{2} v + \frac{1}{4}\right) \pi, \qquad \mu = 4v^2,$$

$$P(v, x) = 1 - \frac{(\mu-1)(\mu-9)}{2!(8x)^2} + \frac{(\mu-1)(\mu-9)(\mu-25)(\mu-49)}{4!(8x)^4} - \cdots$$

and

$$\mathcal{Q}(v, x) = \frac{\mu-1}{8x} - \frac{(\mu-1)(\mu-9)(\mu-25)}{3!(8x)^3} + \cdots.$$

LEMMA 8 (Gradsteyn and Ryzhik, 1980, p. 986, Eq. 8.576). *If $\mu \pm v$ is not a positive odd integer*, *then*

$$S_{\mu,v}(x) = x^{\mu-1} \sum_{m=0}^{p-1} \frac{(-1)^m \Gamma(\frac{1}{2} - \frac{1}{2}\mu + \frac{1}{2}v + m)\, \Gamma(\frac{1}{2} - \frac{1}{2}\mu - \frac{1}{2}v + m)}{(x/2)^m \Gamma(\frac{1}{2} - \frac{1}{2}\mu + \frac{1}{2}v)\, \Gamma(\frac{1}{2} - \frac{1}{2}\mu - \frac{1}{2}v)}$$

$$+ O(x^{\mu-2p}).$$

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